p216 Q1 2016 Fall

1 — $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \xrightarrow{T} \{\vec{e}\,'_1, \vec{e}\,'_2, \vec{e}\,'_3\}$ $\qquad T\vec{e}_i = \sum_{j=1}^{3} T_{ji}\vec{e}_j$

$\vec{e}\,'_1 = \vec{e}_1 + \vec{e}_3 = T\vec{e}_1 = T_{11}\vec{e}_1 + T_{21}\vec{e}_2 + T_{31}\vec{e}_3 \rightarrow T_{11} = 1,\ T_{21} = 0,\ T_{31} = 1$

$\vec{e}\,'_2 = 2\vec{e}_1 + \vec{e}_2 = T\vec{e}_2 \rightarrow T_{12}\vec{e}_1 + T_{22}\vec{e}_2 + T_{32}\vec{e}_3 \rightarrow T_{12} = 2,\ T_{22} = 1,\ T_{32} = 0$

$\vec{e}\,'_3 = 3\vec{e}_2 + \vec{e}_3 = T\vec{e}_3 = T_{13}\vec{e}_1 + T_{23}\vec{e}_2 + T_{33}\vec{e}_3 \rightarrow T_{13} = 0,\ T_{23} = 3,\ T_{33} = 1$

$$\therefore T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

9 points (1 point for each T_{ij})

$\det T = 1(1-0) - 2(0-3) = 1 + 6 = 7 \neq 0$ $\qquad$ ($3\frac{1}{2}$

$\therefore \{\vec{e}\,'_1, \vec{e}\,'_2, \vec{e}\,'_3\}$ is a basis

2. $\vec{u} = (1, 3-i, 2+i)$

$\vec{v} = (3, 1+i, 2i)$

$(\vec{u}, \vec{u}) = (1)^*(1) + (3-i)^*(3-i) + (2+i)^*(2+i)$

$= 1 + (9+1) + (4+1) = 16$ 3 points

$(\vec{v}, \vec{v}) = (3)^*(3) + (1+i)^*(1+i) + (2i)^*(2i)$

$= 9 + 2 + 4 = 15$ 3 points

$(\vec{u}, \vec{v}) = (1)^*(3) + (3-i)^*(1+i) + (2+i)^*(2i)$

$= 3 + (3 - 1 + 4i) + (4i + 2)$

$= 7 + 8i$ (3 p)

$(\vec{v}, \vec{u}) = (\vec{u}, \vec{v})^* = 7 - 8i$ (3 p) $\|(\vec{u}, \vec{v})\| = \sqrt{7^2 + 8^2} = \sqrt{113} < \sqrt{16}\sqrt{15}$ ($\frac{1}{2}$ point)

3. $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$

$\det A = 2(-20+2) - 3(-2) - 4(4)$ 1 point

$= -36 + 6 - 16 = -46$

$\operatorname{adj}(2) = \begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18$

$$\text{adj}(3) = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2$$

$$\text{adj}(-4) = \begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$$

$$\text{adj}(0) = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -11$$

$$\text{adj}(-4) = \begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 14$$

$$\text{adj}(2) = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -5$$

$$\text{adj}(1) = \begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = -10$$

$$\text{adj}(5) = \begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8$$

9 points

$$\therefore C_{ij} = \begin{pmatrix} -18 & 2 & 4 \\ -11 & 14 & -5 \\ -10 & -4 & -8 \end{pmatrix}.$$

$$A^{-1} = \frac{C^T}{\det A} = -\frac{1}{46}\begin{pmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & -5 & -8 \end{pmatrix}$$

→ $1\frac{1}{2}$ points

4. $A = \begin{pmatrix} 1 & 2-i\sqrt{2} \\ 2+i\sqrt{2} & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2-i\sqrt{2} \\ 2+i\sqrt{2} & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - (4+2)$$

$$= \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1) = 0$$

(3 points)

$\therefore \lambda_1 = -1, \quad \lambda_2 = 4$. Let $\vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$$(A - \lambda_1 I)\vec{v} = \begin{pmatrix} 1-\lambda_1 & 2-i\sqrt{2} \\ 2+i\sqrt{2} & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 & 2-i\sqrt{2} \\ 2+i\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$2a + (2+i\sqrt{2})b = 0 \to b = -\frac{2}{2+i\sqrt{2}}a = -\frac{2(2-i\sqrt{2})}{6}a = -\frac{1}{3}\left(1-\frac{i\sqrt{2}}{2}\right)a$$

$$\therefore \vec{v}_1 = a\begin{pmatrix}1\\ -\frac{1}{3}\left(1-i\frac{\sqrt{2}}{2}\right)\end{pmatrix} \qquad \|\vec{v}_1\|^2 = a^2\left(1+\frac{1}{9}\left(1+\frac{1}{2}\right)\right) = a^2\left(\frac{7}{6}\right) = 1$$

$$\therefore a = \sqrt{\frac{6}{7}} \; ; \; \vec{v}_1 = \sqrt{\frac{6}{7}}\begin{pmatrix}1\\ -\frac{1}{3}\left(1-\frac{i\sqrt{2}}{2}\right)\end{pmatrix}$$

3 points

$$(A-\lambda_2 I)\vec{v}_2 = \begin{pmatrix}1-\lambda_2 & 2-i\sqrt{2}\\ 2+i\sqrt{2} & 2-\lambda_2\end{pmatrix}\vec{v}_2 = \begin{pmatrix}-3 & 2-i\sqrt{2}\\ 2+i\sqrt{2} & -2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = 0$$

$$-3a + (2-i\sqrt{2})b = 0 \to b = \frac{3}{2-i\sqrt{2}}a = \frac{3(2+i\sqrt{2})}{6}a = \left(1+\frac{i\sqrt{2}}{2}\right)a$$

$$\therefore \vec{v}_2 = a\begin{pmatrix}1\\ 1+\frac{i\sqrt{2}}{2}\end{pmatrix} \; ; \; \|\vec{v}_2\|^2 = a^2\left(1+1+\frac{1}{2}\right) = \frac{5}{2}a^2 = 1$$

$$\therefore \vec{v}_2 = \sqrt{\frac{2}{5}}\begin{pmatrix}1\\ 1+\frac{i\sqrt{2}}{2}\end{pmatrix}$$

3 points

$$P = [\vec{v}_1, \vec{v}_2] = \begin{pmatrix}\sqrt{\frac{6}{7}} & \sqrt{\frac{2}{5}}\\ -\frac{1}{3}\left(1-\frac{i\sqrt{2}}{2}\right)\sqrt{\frac{6}{7}} & \sqrt{\frac{2}{5}}\left(1+\frac{i\sqrt{2}}{2}\right)\end{pmatrix}$$

$3\frac{1}{2}$ points

5. $[L_1, L_2]f = (L_1 L_2 - L_2 L_1)f$

$L_2 f = -i\left(x^3 \frac{\partial f}{\partial x^1} - x^1 \frac{\partial f}{\partial x^3}\right),$

$$L_1 L_2 f = -i \left(x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2} \right)(-i)\left(x^3 \frac{\partial f}{\partial x^1} - x^1 \frac{\partial f}{\partial x^3} \right)$$

$$= -\left\{ x^2 \frac{\partial f}{\partial x^1} + x^2 x^3 \frac{\partial^2 f}{\partial x^3 \partial x^1} - x^2 x^1 \frac{\partial^2 f}{(\partial x^3)^2} \right.$$

$$\left. - (x^3)^2 \frac{\partial^2 f}{\partial x^2 \partial x^1} + x^3 x^1 \frac{\partial^2 f}{\partial x^2 \partial x^3} \right\} \rightarrow 5 \text{ points}$$

$$L_1 f = -i \left(x^2 \frac{\partial f}{\partial x^3} - x^3 \frac{f}{\partial x^2} \right)$$

$$L_2 L_1 f = -i \left(x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3} \right)(-i)\left(x^2 \frac{\partial f}{\partial x^3} - x^3 \frac{\partial f}{\partial x^2} \right)$$

$$= -\left\{ x^3 x^2 \frac{\partial^2 f}{\partial x^1 \partial x^3} - (x^3)^2 \frac{\partial^2 f}{\partial x^1 \partial x^2} - x^1 x^2 \frac{\partial^2 f}{(\partial x^3)^2} + x^1 \frac{\partial f}{\partial x^2} + x^1 x^3 \frac{\partial^2 f}{(\partial x^3)^2} \right\}$$

$\rightarrow$ 5 point

$$\therefore [L_1, L_2] f = \left(x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \right) f = i L_3 f \rightarrow 2\tfrac{1}{2} \text{ points}$$

6. $\vec{L} = -i \vec{r} \times \vec{\nabla}$

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\theta} \frac{\partial}{\partial \theta} + \frac{1}{r \sin\theta} \hat{\varphi} \frac{\partial}{\partial \varphi}$$

$$\vec{r} \times \vec{\nabla} = r \hat{r} \times \vec{\nabla} = \hat{r} \times \hat{\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin\theta} \hat{r} \times \hat{\varphi} \frac{\partial}{\partial \varphi}$$

$$= \hat{\varphi} \frac{\partial}{\partial \theta} - \frac{1}{\sin\theta} \hat{\theta} \frac{\partial}{\partial \varphi}$$

$$\therefore \vec{L} = i \left(\frac{1}{\sin\theta} \hat{\theta} \frac{\partial}{\partial \varphi} - \hat{\varphi} \frac{\partial}{\partial \theta} \right) \quad 5 \text{ points}$$

$$-\frac{i}{r^2}\vec{r}\times\vec{L} = -\frac{i}{r}\hat{r}\times\vec{L} = \frac{1}{r}\left(\frac{1}{\sin\theta}\hat{r}\times\hat{\theta}\frac{\partial}{\partial\varphi} - \hat{r}\times\hat{\varphi}\frac{\partial}{\partial\theta}\right)$$

$$= \frac{1}{r}\left(\frac{1}{\sin\theta}\hat{\varphi}\frac{\partial}{\partial\varphi} + \hat{\theta}\frac{\partial}{\partial\theta}\right)$$

5 point

$$\therefore\ \vec{\nabla} = \hat{r}\frac{\partial}{\partial r} - \frac{i}{r^2}\vec{r}\times\vec{L}$$

$-2\frac{1}{2}$

7. $$\vec{V} = \frac{x}{\sqrt{x^2-y^2}}\vec{i} - \frac{y}{\sqrt{x^2-y^2}}\vec{j} \qquad (x,y)\neq 0$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ \frac{x}{\sqrt{x^2-y^2}} & \frac{-y}{\sqrt{x^2-y^2}} & 0 \end{vmatrix} = \vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\left\{\frac{\partial}{\partial x}\left(\frac{-y}{\sqrt{x^2-y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2-y^2}}\right)\right.$$

$$= \vec{k}\left(\frac{xy}{(x^2-y^2)^{3/2}} - \frac{xy}{(x^2-y^2)^{3/2}}\right) = 0 \qquad x\neq y$$

$$\therefore\ \vec{V} = \nabla\phi$$

6 points

$$\int_C \vec{V}\cdot d\vec{r} = \int_C V_x\,dx + V_y\,dy = \int_C \frac{1}{\sqrt{x^2-y^2}}(x\,dx - y\,dy)$$

let $r^2 = x^2 - y^2$ then $2r\,dr = 2x\,dx - 2y\,dy$

or $x\,dx - y\,dy = r\,dr \Rightarrow \int_C \vec{V}\cdot d\vec{r} = \int_C \frac{r\,dr}{r} = \int_0^{\sqrt{x^2-y^2}} dr = r\Big|_1^{\sqrt{x^2-y^2}}$

$$= \sqrt{x^2-y^2} - 1$$

5 point

$\phi(x,y) - \phi(1,0) = \sqrt{x^2-y^2} - 1$ $\frac{1}{2}$ point

$\phi(x,y) = \sqrt{x^2-y^2} + \text{const}$

8. $\vec{V} = xy\,\hat{i} + z^2\,\hat{j} + 2yz\,\hat{k}$

$$\oint_S \vec{V}\cdot d\vec{a} = \int_{6\ \text{faces}} \vec{V}\cdot d\vec{a}$$

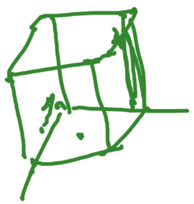
Face 1 → $z=0$, $\vec{V}\cdot d\vec{a} = -\vec{V}\cdot\hat{k}\,dx\,dy = -2yz\,dx\,dy$ 1 points for each face

$$\int_{\text{Face 1}} \vec{V}\cdot d\vec{a} = 0$$

Face 2 $z=1$ $\vec{V}\cdot d\vec{a} = \vec{V}\cdot\hat{k}\,dx\,dy = 2yz\,dx\,dy = 2y\,dx\,dy$

$$\int_{\text{Face 2}} \vec{V}\cdot d\vec{a} = 2\int_0^1 dx \int_0^1 y\,dy = 1$$

Face 3



$y=0$ $(dx\,dz)$ $\vec{V}\cdot d\vec{a} = -\vec{V}\cdot\hat{j}\,dx\,dz = -z^2\,dx\,dz$

$$\int_{(3)} \vec{V}\cdot d\vec{a} = -\int_0^1 z^2\,dz \int_0^1 dx = -\frac{z^3}{3}\Big|_0^1 = -\frac{1}{3}$$

$y=1$ Face 4 → $\vec{V}\cdot d\vec{a} = \vec{V}\cdot\hat{j}\,dx\,dz = z^2\,dx\,dz$

$$\therefore \int_{(4)} \vec{v}\cdot d\vec{a} = \int_0^1 z^2\,dz \int_0^1 dx = \frac{1}{3}$$

Face 5 : $x=0$ $\quad \vec{v}\cdot d\vec{a} = -\vec{v}\cdot\hat{i}\,dy\,dz = -xy\,dy\,dz = 0$ $\quad$ 2p

$$\int_{(5)} \vec{v}\cdot d\vec{a} = 0$$

Face 6: $x=1$ $\quad \vec{v}\cdot d\vec{a} = \vec{v}\cdot\hat{i}\,dy\,dz = xy\,dy\,dz = y\,dy\,dz$ $\quad$ 2p

$$\int \vec{v}\cdot d\vec{a} = \int_0^1 y\,dy \int_0^1 dz = \frac{1}{2}$$

$$\therefore \oint_S \vec{v}\cdot d\vec{a} = 0 + 1 + \left(-\frac{1}{3}\right) + \left(\frac{1}{3}\right) + (0) + \left(\frac{1}{2}\right) = \frac{3}{2}$$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(2yz)$$

$$= y + 2y = 3y$$ 3p

$$\therefore \int_V \vec{\nabla}\cdot\vec{v}\,d^3x = \int_0^1 dx \int_0^1 dz \int_0^1 3y\,dy = \frac{3}{2}$$ 3p

$$\therefore \int (\vec{\nabla}\cdot\vec{v})\,d^3x = \oint_{\text{cube}} \vec{v}\cdot d\vec{a}$$ $\frac{1}{2}$

9. $\vec{v} = y^2\,\vec{i} + xy\,\vec{j} + xz\,\vec{k}$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = \vec{i}\,(0) - \vec{j}\,(z) + \vec{k}\,(y - 2y)$$

$$= -z\,\vec{j} - y\,\vec{k} \qquad \text{o.} \quad 3$$

$z = R^2 - x^2 - y^2 \geq 0$ is the upper hemisphere

$$d\vec{a} = R^2 \sin\theta\, d\theta\, d\varphi\ \hat{r} \qquad 2$$

$$= R^2 \sin\theta\, d\theta\, d\varphi\ (\sin\theta\,(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}) + \cos\theta\,\vec{k})$$

$$(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = R^2 \sin\theta\, d\theta\, d\varphi\ (-z\sin\varphi\sin\theta - y\cos\theta)$$

$$= -R^3 \sin\theta\ (\sin\theta\cos\theta\sin\varphi + \sin\theta\cos\theta\sin\varphi)$$

$$= -2R^2 \sin^2\theta\cos\theta\sin\varphi$$

$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = -2R^2 \int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta \int_0^{2\pi} \sin\varphi = 0 \qquad 3$$

$x = R\cos\varphi,\ y = R\sin\varphi$

$$\oint_C \vec{v}\cdot d\vec{l} = \int_C y^2 dx + xy\, dy + xz\, dz \quad (z \to 0)$$

$$= R^2 \int_0^{2\pi} (\sin^2\varphi(-\sin\varphi) + \sin\varphi\cos\varphi(\cos\varphi))\, d\varphi = 0$$

periodic function. $4\frac{1}{2}$

10. $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) \equiv \vec{\nabla}\cdot\vec{C}$ where $\vec{C} = \vec{A}\times\vec{B}$

$\vec{\nabla}\cdot\vec{C} = \partial_1 C_1 + \partial_2 C_2 + \partial_3 C_3$

$C_1 = (A_2 B_3 - A_3 B_2)$, $C_2 = (A_3 B_1 - A_1 B_3)$, $C_3 = (A_1 B_2 - A_2 B_1)$

$\therefore\ \vec{\nabla}\cdot\vec{C} = \partial_1(A_2 B_3 - A_3 B_2) + \partial_2(A_3 B_1 - A_1 B_3) + \partial_3(A_1 B_2 - A_2 B_1)$

$= \partial_1 A_2 B_3 - \partial_1 A_3 B_2 + \partial_2 A_3 B_1 - \partial_2 A_1 B_3 + \partial_3 A_1 B_2 - \partial_3 A_2 B_1$

$+ A_2 \partial_1 B_3 - A_3 \partial_1 B_2 + A_3 \partial_2 B_1 - A_1 \partial_2 B_3 + A_1 \partial_3 B_2 - A_2 \partial_3 B_1$

6p.

$= (\partial_1 A_2 - \partial_2 A_1) B_3 + (-\partial_1 A_3 + \partial_3 A_1) B_2 + (\partial_2 A_3 - \partial_3 A_2) B_1$

$+ A_2(\partial_1 B_3 - \partial_3 B_1) + A_3(-\partial_1 B_2 + \partial_2 B_1) + A_1(\partial_3 B_2 - \partial_2 B_3)$

3p $(\vec{\nabla}\times\vec{A})\cdot\vec{B} = (\vec{\nabla}\times\vec{A})_1 B_1 + (\vec{\nabla}\times\vec{A})_2 B_2 + (\vec{\nabla}\times\vec{A})_3 B_3$

$= (\partial_2 A_3 - \partial_3 A_2) B_1 + (\partial_3 A_1 - \partial_1 A_3) B_2 + (\partial_1 A_2 - \partial_2 A_1) B_3$

3p $(\vec{\nabla}\times\vec{B})\cdot\vec{A} = (\vec{\nabla}\times\vec{B})_1 A_1 + (\vec{\nabla}\times\vec{B})_2 A_2 + (\vec{\nabla}\times\vec{B})_3 A_3$

$= (\partial_2 B_3 - \partial_3 B_2) A_1 + (\partial_3 B_1 - \partial_1 B_3) A_2 + (\partial_1 B_2 - \partial_2 B_1) A_3$

2p $\therefore\ \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - (\vec{\nabla}\times\vec{B})\cdot\vec{A}$